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Vertex operator of $U_q(B_l^{(1)})$ for level one

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Abstract. In this paper, we give the explicit formulae of vertex operators of $U_q(B_l^{(1)})$ for level one as operators on the Fock space. Meanwhile, we point out that the free-field realization (by one fermionic field and l bosonic fields) of highest weight module with highest weight Λ_l has two irreducible modules.

1. Introduction

One of the central subjects of mathematical physics has been the studies on exactly solvable models in two dimensions for many years. The central problem is the determination of eigenstate and correlation functions (form factors) in exactly solvable models. In [3, 7] a new scheme was given for solving the six-vertex model and associated XXZ chain in the antiferromagnetic regime using the newly discovered quantum affine symmetry of the system. The approach of that paper has been extended to higher spin chains [8–11], to the higher rank case [1, 18] and to ABF models [14]. The analogous approach in integrable massive field theory also has been developed by Lukyanov [15]. All of these papers are concerned with models constructed on the quantum affine algebra $U_q(\widehat{A}_l)$. The key object in this approach is vertex operators which have first been introduced by Frenkel and Reshetikhin [2]. In order to extend their scheme to the models with symmetry of $U_q(B_l^{(1)})$, we construct the q-vertex operators related to $U_q(B_l^{(1)})$ in this paper. The physical application will be developed elsewhere. In section 2, we briefly recall the free-field realization of $U_q(B_l^{(1)})$. In section 3, we deduce the vertex operators.

2. Free-field realization of $U_q(B_l^{(1)})$

In this section we briefly recall the free-field realization of $U_q(B_l^{(1)})$ [4].

2.1. Definition of $U_q(B_l^{(1)})$

First, fix some notation that we will use. Let $\widehat{P} = \boldsymbol{Z}\Lambda_0 \oplus \boldsymbol{Z}\Lambda_1 \oplus \cdots \oplus \boldsymbol{Z}\Lambda_l \oplus \boldsymbol{Z}\delta$ be the weight lattice of $U_q(B_l^{(1)})$ and let $\alpha_i = \sum_{j=0}^{l} a_{ji}\Lambda_j + \delta_{i,0}\delta$ $(i = 0, 1, \ldots, l)$ be simple roots.

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So we have $\delta = \alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_l)$, $\widehat{Q} = \boldsymbol{Z}\alpha_0 \oplus \boldsymbol{Z}\alpha_1 \oplus \cdots \oplus \boldsymbol{Z}\alpha_l$ being its root lattice. The symmetric bilinear form (,) on $\widehat{P}$ is defined by

$$(\Lambda_0, \Lambda_i) = 0 \qquad (\Lambda_0, \delta) = 1 \qquad (\Lambda_i, \alpha_j) = d_i \delta_{ij} \qquad i, j = 0, 1, \ldots, l. \tag{2.1}$$

We can get

$$\begin{cases} (\Lambda_1, \delta) = 1 \qquad (\delta, \delta) = 0 \qquad (\Lambda_l, \delta) = 1 \\ (\Lambda_i, \Lambda_j) = d_i (\overline{A}^{-1})_{ij} \qquad 1 \leqslant i, j \leqslant l \qquad (\Lambda_i, \delta) = 2 \qquad 1 < i < l \\ (\Lambda_i, \delta) = 2 \qquad 1 \leqslant i \leqslant l \qquad (\alpha_i, \alpha_j) = d_i a_{ij} \end{cases} \tag{2.2}$$

where $A = (a_{ij})_{i,j=0}^{l}$ is the Cartan matrix of $U_q(B_l^{(1)})$ and $\overline{A}$ is the matrix of A removed by the first line and first column.

For $U_q(\widehat{B}_2)$,

$$d_0 = d_1 = 2d_2 = 1$$
$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -2 & -2 & 2 \end{bmatrix}$$

and for $U_q(\widehat{B}_l)$ $(l \geqslant 3)$,

$$d_0 = d_1 = \cdots = d_{l-1} = 2d_l = 1$$
$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix}.$$

Define the dual space of $\widehat{P}$ as $\widehat{P^*} = \boldsymbol{Z}h_0 \oplus \boldsymbol{Z}h_1 \oplus \cdots \oplus \boldsymbol{Z}h_l \oplus \boldsymbol{Z}d$. The dual pairing $\langle , \rangle$ is defined by

$$\begin{cases} \langle h_i, \lambda \rangle := d_i^{-1}(\alpha_i, \lambda) \qquad \lambda \in \widehat{P} \\ \langle d, \lambda \rangle := (\Lambda_0, \lambda) \qquad \lambda \in \widehat{P}. \end{cases} \tag{2.3}$$

Later we will use the weight lattice and the root lattice of $U_q(B_l)$. The weight lattice $P = \boldsymbol{Z}\lambda_1 \oplus \boldsymbol{Z}\lambda_2 \oplus \cdots \oplus \boldsymbol{Z}\lambda_l$, $\lambda_i = \Lambda_i - a_i^\vee \Lambda_0$, where $a_0^\vee = a_1^\vee = a_l^\vee = 1$, $a_2^\vee = \cdots = a_{l-1}^\vee = 2$. The root lattice of $U_q(B_l)$ is $Q = \boldsymbol{Z}\alpha_1 \oplus \boldsymbol{Z}\alpha_2 \oplus \cdots \oplus \boldsymbol{Z}\alpha_l$. Note that $\lambda_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_l$ and $2\lambda_l = \alpha_1 + 2\alpha_2 + \cdots + l\alpha_l$, so the weight lattice can be spanned by $\alpha_2, \alpha_3, \ldots, \alpha_l, \lambda_l$. In this article, we assume $-1 < q < 0$ and use the following standard notation,

$$\omega = \frac{1}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})} \qquad q_i = q^{d_i}$$
$$[m]_{q_i} = \frac{{q_i}^m - {q_i}^{-m}}{q_i - {q_i}^{-1}} \qquad m \in \boldsymbol{Z}$$
$$\begin{bmatrix} n \\ m \end{bmatrix}_{q_i} = \frac{[1]_{q_i}[2]_{q_i} \cdots [n]_{q_i}}{[1]_{q_i}[2]_{q_i} \cdots [m]_{q_i}[1]_{q_i}[2]_{q_i} \cdots [n-m]_{q_i}}$$

and when $q_i = q$ we omit the index. The quantum affine algebra $U_q(B_l^{(1)})$ is an associative algebra over $\mathcal{C}$ with unity generated by e_i, f_i, $q_i^{\pm h_i} (i = 0, 1, 2, \ldots, l)$ and $q^{\pm d}$. $U_q'(B_l^{(1)})$

is the subalgebra of $U_q(\widehat{B_l^{(1)}})$ generated by $\{q_i^{\pm h_i}, e_i, f_i \mid 0 \leqslant i \leqslant l\}$. The defining relations are as follows ($\forall h, h' \in P^*$):

$$\begin{cases} q^h q^{h'} = q^{h+h'} \\ q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i \\ q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i \\ [e_i, f_j] = \delta_{i,j} \dfrac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} \\ \displaystyle\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} {e_i}^{1-a_{ij}-n} e_j {e_i}^n = 0 \qquad i \neq j \\ \displaystyle\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} {f_i}^{1-a_{ij}-n} f_j {f_i}^n = 0 \qquad i \neq j. \end{cases} \tag{2.4}$$

The algebra $U_q(B_l^{(1)})$ has a Hopf algebra structure with the following coproduct Δ: $U_q(B_l^{(1)}) \to U_q(B_l^{(1)}) \otimes U_q(B_l^{(1)})$

$$\begin{cases} \Delta(e_i) = e_i \otimes 1 + q_i^{h_i} \otimes e_i \qquad \Delta(f_i) = f_i \otimes q_i^{-h_i} + 1 \otimes f_i \\ \Delta(q^h) = q^h \otimes q^h \qquad h \in P^*. \end{cases} \tag{2.5}$$

As above this algebra is defined through Chevalley generators.

2.2. *Drinfeld realization [6]*

For free-field realization, $U_q(B_l^{(1)})$ defined by Chevalley generators is not convenient. Drinfeld (1988) [6] has given another realization. The new realization is an associative algebra generated by the elements $\{x_i^{\pm}(n), a_i(m), \gamma^{\pm\frac{1}{2}}, q_i^{h_i} \mid 1 \leqslant i \leqslant l, m \in \mathbf{Z}\backslash\{0\}, n \in \mathbf{Z}\}$ satisfying the following relations:

$$\begin{cases} \gamma^{\pm\frac{1}{2}} \text{ is the centre of the algebra} \\ [a_i(n), a_j(m)] = \delta_{n+m,0} \dfrac{1}{n} [na_{ij}]_{q_i} \dfrac{\gamma^n - \gamma^{-n}}{q_j - q_j^{-1}} \\ q_i^{h_i} x_j^{\pm}(n) q_i^{-h_i} = {q_i}^{\pm a_{ij}} {x_j}^{\pm}(n) \\ [a_i(n), {x_j}^{\pm}(m)] = \pm \dfrac{1}{n} [na_{ij}]_{q_i} \gamma^{\mp|n|/2} {x_j}^{\pm}(n+m) \\ {x_i}^{\pm}(n+1) {x_j}^{\pm}(m) - {q_i}^{\pm a_{ij}} {x_j}^{\pm}(m) {x_i}^{\pm}(n+1) \\ \qquad = {q_i}^{\pm a_{ij}} {x_i}^{\pm}(n) {x_j}^{\pm}(m+1) - {x_j}^{\pm}(m+1) {x_i}^{\pm}(n) \\ [{x_i}^{+}(n), {x_j}^{-}(m)] = \delta_{i,j} \dfrac{\gamma^{(n-m)/2} {\varphi_i}^{+}(n+m) - \gamma^{-(n-m)/2} {\varphi_i}^{-}(n+m)}{q_i - {q_i}^{-1}} \\ \displaystyle\sum_{\pi \in \sum_p} \sum_{k=0}^{p} (-1)^k \begin{bmatrix} p \\ k \end{bmatrix}_{q_i} {x_i}^{\pm}(r_{\pi(1)}) \cdots {x_i}^{\pm}(r_{\pi(k)}) {x_j}^{\pm}(s) {x_i}^{\pm}(r_{\pi(k+1)}) \cdots {x_j}^{\pm}(r_{\pi(p)}) = 0 \end{cases} \tag{2.6}$$

if $i \neq j$, for sequences of integers $r_1, \ldots, r_p$. Where $p = 1 - a_{ij}$, $\sum_p$ is the symmetric group on p letters, and the ${\varphi_i}^{\pm}(r)$ are determined by equating powers of u in the formal power series

$$\sum_{r=0}^{\infty} \varphi_i{}^{\pm}(r) u^{\mp r} = q_i^{\pm h_i} \exp \pm (q_i - q_i{}^{-1}) \sum_{n=1}^{\infty} a_i(\pm n) u^{\mp n}. \tag{2.7}$$

Define maps $w_i{}^{\pm}$: $U_q(\widehat{g}) \to U_q(\widehat{g})$ by

$$w_i{}^{\pm} a = x_i{}^{\pm}(0) a - q_i^{\pm h_i} a q_i^{\mp h_i} x_i{}^{\pm}(0) \tag{2.8}$$

then, the isomorphism between the two realizations is

$$\begin{cases} q^{h_0} = \gamma q^{-h_1 - 2h_2 - \cdots - 2h_{l-1} - h_l} \qquad e_i = x_i{}^+(0) \qquad f_i = x_i{}^-(0) \\ \qquad \text{for } i = 1, 2, \ldots, l \\ e_0 = w^-{}_2 w^-{}_3 \cdots w^-{}_l w^-{}_l \cdots w^-{}_2 x_1{}^-(1) q^{h_0} \gamma^{-1} = e_0' q^{h_0} \gamma^{-1} \\ f_0 = q^{2l-3} \omega^2 q^{-h_0} \gamma w^+{}_2 w^+{}_3 \cdots w^+{}_l w^+{}_l \cdots w^+{}_2 x_1{}^+(-1) = q^{-h_0} \gamma f_0'. \end{cases} \tag{2.9}$$

2.3. *Central extension of weight lattice*

It is enough to consider only the central extension of the root lattice Q to construct the representation of $U_q(B_l^{(1)})$ and the vertex operators for our case. However, for the general construction of the vertex operators it is necessary to define the central extension of weight lattice P.

Define the group algebra $C[P]$ generated by the symbols $\{e^{\alpha_2}, \ldots, e^{\alpha_{l-1}}, e^{\alpha_l}, e^{\lambda_l}\}$ and satisfying the following relations:

$$\begin{cases} e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \qquad 2 \leqslant i \neq j \leqslant l \\ e^{\lambda_l} e^{\alpha_i} = (e^{I\pi/2})^{(l-1)\delta_{i,l}} e^{\alpha_i} e^{\lambda_l} \qquad 2 \leqslant i \leqslant l. \end{cases} \tag{2.10}$$

For $\alpha = m_2\alpha_2 + \cdots + m_{l-1}\alpha_{l-1} + m_l\alpha_l + m_1\lambda_l$, we denote $e^{m_2\alpha_2} \cdots e^{m_{l-1}\alpha_{l-1}} e^{m_l\alpha_l} e^{m_1\lambda_l}$ by e^{α}. So a simple calculation shows

$$\begin{cases} e^{\lambda_l} e^{\alpha_1} = (e^{I\pi/2})^{(l-1)l} e^{\alpha_1} e^{\lambda_l} \qquad e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \qquad 1 \leqslant i \neq j \leqslant l \\ e^{\lambda_1} e^{\alpha_l} = -e^{\alpha_l} e^{\lambda_1} \qquad e^{\lambda_1} e^{\alpha_1} = -e^{\alpha_1} e^{\lambda_1}. \end{cases} \tag{2.11}$$

Remark. From our definition $e^{-\alpha} e^{\alpha} \neq 1$, for example $\alpha = \alpha_l + \lambda_l$, $e^{-\alpha} e^{\alpha} = e^{-\alpha_l} e^{-\lambda_l} e^{\alpha_l} e^{\lambda_l} = e^{-I\pi(l-1)/2}$. In this section I is an imaginary unit.

2.4. *Level-one module*

Bernard [4] has given the free-field realization of the level-one module by using one fermionic field and l bosonic fields. Here, we reconstruct the three-level-one module and point out a fact which has previously been overlooked.

Let

$$H_{\mathcal{NS}} := C[a_j(-m), \Psi(-k) \mid 1 \leqslant j \leqslant l, m \in \boldsymbol{Z}_{>0}, k \in \boldsymbol{Z}_{\geqslant 0} + \tfrac{1}{2}] \otimes C[Q]$$

in $\mathcal{NS}$ cases.

$$H_{\mathcal{R}} := C[a_j(-m), \Psi(-k) \mid 1 \leqslant j \leqslant l, m \in \boldsymbol{Z}_{>0}, k \in \boldsymbol{Z}_{\geqslant 0}] \otimes C[Q] e^{\lambda_l}$$

in $\mathcal{R}$ cases.

We define the operators $a_i(k)$ $(1 \leqslant i \leqslant l)$, $\Psi(k)$, ∂_λ, e^{λ} for $\lambda \in P$ and d on $H_{\mathcal{NS}}$ and $H_{\mathcal{R}}$ as follows.

For $f \otimes e^{\beta} = a_{i_1}(-n_1)\cdots a_{i_k}(-n_k)\Psi(-k_1)\cdots\Psi(-k_n)\otimes e^{\beta} \in H_{\mathcal{NS}}$ and $f \otimes e^{\beta} e^{\lambda_l} \in H_{\mathcal{R}}$,

$$\begin{cases} a_j(k)\cdot f\otimes e^{\beta} = \begin{cases} a_j(k) f\otimes e^{\beta} & (k<0) \\ [a_j(k), f]\otimes e^{\beta} & (k>0) \end{cases} \\ \Psi(k)\cdot f\otimes e^{\beta} = \begin{cases} \Psi(k) f\otimes e^{\beta} & (k\leqslant 0) \\ cr\{\Psi(k), f\}\otimes e^{\beta} & (k>0) \end{cases} \\ \partial_{\lambda}\cdot f\otimes e^{\beta} = (\lambda,\beta) f\otimes e^{\beta} \qquad \partial_{\lambda}\cdot f\otimes e^{\beta}e^{\lambda_l} = (\lambda, \beta+\lambda_l) f\otimes e^{\beta}e^{\lambda_l} \\ e^{\lambda}\cdot f\otimes e^{\beta} = f\otimes e^{\lambda}e^{\beta} \qquad e^{\lambda}\cdot f\otimes e^{\beta}e^{\lambda_l} = f\otimes e^{\lambda}e^{\beta}e^{\lambda_l} \qquad \text{for } \lambda\in P \\ d\cdot f\otimes e^{\beta} = \left(-\sum_{i=1}^{k} n_i - \sum_{j=1}^{n} k_j - (\beta,\beta)/2 + (\lambda_1,\lambda_1)\frac{1-G}{4}\right) f\otimes e^{\beta} \\ \qquad \text{for the } \mathcal{NS} \text{ case} \\ d\cdot f\otimes e^{\beta}e^{\lambda_l} = \left(-\sum_{i=1}^{k} n_i - \sum_{j=1}^{n} k_j - (\beta,\beta)/2 + (\lambda_l,\lambda_l)/2\right) f\otimes e^{\beta}e^{\lambda_l} \\ \qquad \text{for the } \mathcal{R} \text{ case} \end{cases} \tag{2.12}$$

where $G = 1$ if $f\otimes e^{\beta} \in V(\Lambda_0)$ and $G = -1$ if $f\otimes e^{\beta}\in V(\Lambda_1)$.

Let

$$x_i^{\pm}(z) := \sum_{n\in \boldsymbol{Z}} x_i^{\pm}(n) z^{-n} \qquad (1\leqslant i\leqslant l-1).$$

We define the action of $U_q(B_l^{(1)})$

$$\begin{cases} \gamma \to q \qquad q_j^{h_j} \to q^{\partial_{\alpha_j}} \qquad (1\leqslant j\leqslant l) \\ x_i{}^{\pm}(z) \to z\exp\left(\pm\sum_{k=1}^{\infty}\frac{a_i(-k)}{[k]}q^{\mp k/2}z^{k}\right)\exp\left(\mp\sum_{k=1}^{\infty}\frac{a_i(k)}{[k]}q^{\mp k/2}z^{-k}\right)e^{\pm\alpha_i}z^{\pm\partial_{\alpha_i}} \\ x_l{}^{\pm}(z) \to z^{\frac{1}{2}}\exp\left(\pm\sum_{k=1}^{\infty}\frac{a_l(-k)}{[k]}q^{\mp k/2}\omega z^{k}\right)\exp\left(\mp\sum_{k=1}^{\infty}\frac{a_l(k)}{[k]}q^{\mp k/2}\omega z^{-k}\right) \\ \qquad \times e^{\pm\alpha_l}z^{\pm\partial_{\alpha_l}}\Psi(z) \end{cases} \tag{2.13}$$

where

$$\Psi(z) = \sum_{n}\Psi(n)z^{-n}$$

with $n\in \boldsymbol{Z}+\frac{1}{2}$(or $n\in\boldsymbol{Z}$) in the $\mathcal{NS}$ (or $\mathcal{R}$) cases, respectively, and

$$\begin{cases} \{\Psi(n),\Psi(m)\} = (q^{n}+q^{-n})\delta_{n+m,0} \\ [a_i(n), a_j(m)] = \delta_{n+m,0}\frac{1}{n}[na_{ij}]_{q_i}\frac{q^{n}-q^{-n}}{q_j-q_j^{-1}} \\ [a_i(n),\Psi(m)] = 0. \end{cases} \tag{2.14}$$

Define

$$G = (-1)^{2\sum_{i=1}^{l}\partial_{\lambda_i} + N_F} \tag{2.15}$$

for $\mathcal{NS}$ cases,

$$G = (-1)^{2\sum_{i=1}^{l}\partial_{\lambda_i} + N_F - 2\sum_{i=1}^{l}(\lambda_i, \lambda_l)} \tag{2.16}$$

for $\mathcal{R}$ cases, where N_F denotes the fermion's number operator. We easily find that G commutes with all the elements of $U_q(B_l^{(1)})$. Through the eigenvalues of G, we can divide the Fock space into irreducible ones, i.e. four irreducible $U_q(B_l^{(1)})$ modules, $V(\Lambda_0), V(\Lambda_1), V(\Lambda_l), V(\Lambda_l{}')$, whose highest weight vectors are $v_{\Lambda_0} = 1 \otimes 1, v_{\Lambda_1} = 1 \otimes e^{\lambda_1}, v_{\Lambda_l} = 1 \otimes e^{\lambda_1}, v'_{\Lambda_l} = \Psi(0) \otimes e^{\lambda_l}$, respectively. The first two modules are $\mathcal{NS}$ cases and the others are in $\mathcal{R}$ cases.We have not seen that the reducibility in $\mathcal{R}$ cases has been discussed before.

Remark. Here we do not use the Klein operator following Bernard [4], but the realization given above is still due to Bernard.

3. Vertex operators

Let us review the definition and some properties of the vertex operators.

3.1. Finite-dimensional $U'_q(B_l^{(1)})$ module

Let V be a finite-dimensional $U'_q(B_l^{(1)})$ module with basis $\{v_m \mid 1 \leqslant m \leqslant 2l+1\}$, and the representation of e_i, f_i, h_i as follows:

$$\begin{cases} e_i = f_i{}^t = E_{i,i+1} + E_{2l-i+1,2l-i+2} & i \neq 0, l \\ e_l = f_l{}^t = \omega^{-\frac{1}{2}}(E_{l,l+1} + E_{l+1,l+2}) \\ e_0 = f_0{}^t = E_{2l,1} + E_{2l+1,2} \\ h_i = E_{i,i} - E_{i+1,i+1} + E_{2l-i+1,2l-i+1} - E_{2l-i+2,2l-i+2} & i \neq 0, l \\ h_0 = -E_{1,1} - E_{2,2} + E_{2l,2l} + E_{2l+1,2l+1} \\ h_l = 2E_{l,l} - 2E_{l+2,l+2}. \end{cases} \tag{3.1}$$

Define the $U_q(B_l^{(1)})$ module structure on V_z as follows:

$$\begin{cases} e_i(v_m \otimes z^n) = e_i v_m \otimes z^{n+\delta_{i,0}} & f_i(v_m \otimes z^n) = f_i v_m \otimes z^{n-\delta_{i,0}} \\ h_i(v_m \otimes z^n) = h_i v_m \otimes z^n & d(v_m \otimes z^n) = n v_m \otimes z^n. \end{cases} \tag{3.2}$$

We call V_z the affinization of V as a $U_q(B_l^{(1)})$ module of level zero.

3.2. Definition of q-vertex operators

The intertwiners of $U_q(B_l^{(1)})$ modules

$$\widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) : V(\Lambda_i) \to V(\Lambda_j) \otimes V_z$$

are called type-I vertex operators and the operators

$$\widehat{\Psi}_{\Lambda_i}^{V \Lambda_j}(z) : V(\Lambda_i) \to V_z \otimes V(\Lambda_j)$$

are called type-II vertex operators, where $\otimes$ is the tensor product with an appropriate completion. Denote the vertex operators as a formal series

$$\widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) = \sum_{m=1}^{2l+1} \widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(z) \otimes v_m$$
$$\widehat{\Psi}_{\Lambda_i}^{V \Lambda_j}(z) = \sum_{m=1}^{2l+1} v_m \otimes \widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(z)$$
$$\widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(z) = \sum_{n \in \boldsymbol{Z}} \widehat{\Psi}_{\Lambda_i m}^{\Lambda_j V}(n) z^n$$
$$\widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(z) = \sum_{n \in \boldsymbol{Z}} \widehat{\Psi}_{\Lambda_i m}^{V \Lambda_j}(n) z^n.$$

There exist four type-I (respectively, type-II) vertex operators [2, 13]:

$$\widehat{\Psi}_{\Lambda_0}^{\Lambda_1 V}(z) : V(\Lambda_0) \rightarrow V(\Lambda_1) \otimes V_z$$
$$\widehat{\Psi}_{\Lambda_1}^{\Lambda_0 V}(z) : V(\Lambda_1) \rightarrow V(\Lambda_0) \otimes V_z$$
$$\widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z) : V(\Lambda_l) \rightarrow V(\Lambda_l') \otimes V_z$$
$$\widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z) : V(\Lambda_l') \rightarrow V(\Lambda_l) \otimes V_z.$$

We can impose the nomalization condition

$$\widehat{\Psi}_{\Lambda_0}^{\Lambda_1 V}(z) \cdot v_{\Lambda_0} = v_{\Lambda_1} \otimes v_{2l+1} + (\text{terms of positive powers in } z)$$
$$\widehat{\Psi}_{\Lambda_1}^{\Lambda_0 V}(z) \cdot v_{\Lambda_1} = v_{\Lambda_0} \otimes v_1 + (\text{terms of positive powers in } z)$$
$$\widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z) \cdot v_{\Lambda_l} = v_{\Lambda_l} \otimes v_{l+1} + (\text{terms of positive powers in } z)$$
$$\widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z) \cdot v_{\Lambda_l}' = v_{\Lambda_l}' \otimes v_{l+1} + (\text{terms of positive powers in } z).$$

Later we will find that the formulae of $\widehat{\Psi}_{\Lambda_l}^{\Lambda_l' V}(z)$ and $\widehat{\Psi}_{\Lambda_l'}^{\Lambda_l V}(z)$ are identical, so we will denote them as $\widehat{\Psi}_{\Lambda_l}^{\Lambda_l V}(z)$.

3.3. Vertex operator of type-I

From the intertwining relation (we only need to describe the type-I vertex operator, the type-II is quite parallel),

$$\Delta(x) \circ \widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) = \widehat{\Psi}_{\Lambda_i}^{\Lambda_j V}(z) \circ x \qquad (3.3)$$

then we can get the commutation relations between the type-I vertex operator with Chevalley generators. Here we write the relations partially:

$$\begin{cases} qf_2\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}3}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}3}(z)f_2 \\ qf_3\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}4}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}3}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}4}(z)f_3 \\ \vdots \quad \vdots \quad \vdots \\ qf_{l-1}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l-1}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)f_{l-1} \\ f_l\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z)+\omega^{-\frac{1}{2}}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z)f_l \\ qf_l\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z)+\omega^{-\frac{1}{2}}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z)f_l \\ qf_{l-1}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+3}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+3}(z)f_{l-1} \\ \vdots \quad \vdots \quad \vdots \\ qf_2\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l-1}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l}(z)f_2 \\ qf_1\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)+\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)f_1 \\ q^{-1}e'_0\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)+q^2z\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2}(z)=\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)e'_0 \end{cases} \tag{3.4}$$

$$\begin{cases} [e_i,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)]=0 \qquad i=1,\dots,l \\ [e_i,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z)]=0 \qquad j\neq i,2l+1-i,\ i=1,2,\dots,l \\ [e_i,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}i}(z)]=q_i^{h_i}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}i+1}(z) \qquad i=1,2,\dots,l-1 \\ [e_i,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l-i+1}(z)]=q_i^{h_i}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l-i+2}(z) \qquad i=1,2,\dots,l \\ [e_l,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)]=\omega^{-\frac{1}{2}}q_l^{h_l}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z) \\ [e_l,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z)]=\omega^{-\frac{1}{2}}q_l^{h_l}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z) \\ [f'_0,\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)]=0 \end{cases} \tag{3.5}$$

$$\begin{cases} q^{h_i}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z)q^{-h_i}=q\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z) \qquad j=i,\ 2l-i+1,\ i\neq 0,l \\ q^{h_i}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z)q^{-h_i}=q^{-1}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z) \qquad j=i+l,\ 2l-i+2,\ i\neq 0,l \\ [q^{h_i},\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z)]=0 \qquad j\neq i,\ i+1,\ 2l-i+1,\ 2l-i+2,\ i\neq 0,l \\ [q_l^{h_l},\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z)]=0 \qquad j\neq l,\ l+2 \\ q_l^{h_l}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)q_l^{-h_l}=q\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z) \\ q^{h_l}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z)q^{-h_l}=q^{-1}\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z). \end{cases} \tag{3.6}$$

Defining operators $P^{\pm}{}_i$

$$P^{\pm}{}_i x=[x^{\pm}{}_i(0),x]q_i^{\mp h_i} \tag{3.7}$$

we have

$$P^{\pm}{}_i w_j{}^{\mp} x = \frac{(q_j^{\pm h_j} x q_j^{\mp h_j} - q_j^{\mp h_j} x q_j^{\pm h_j})}{q_i - q_i{}^{-1}} \delta_{i,j} + W_j{}^{\mp} P^{\pm}{}_i x. \tag{3.8}$$

Using them, we get

$$P^+{}_2 P^+{}_3 \cdots P^+{}_l P^+{}_l \cdots P^+{}_3 P^+{}_2 e'_0 = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 x^-{}_1(1) \tag{3.9}$$

and the consistent identity

$$\begin{aligned} &q^{2l-2}\omega(\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)) f_1 - q f_1 \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)) \\ &\qquad = (qz)^{-1}(\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) x_1^-(1) - q^{-1} x_1^-(1) \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)). \end{aligned} \tag{3.10}$$

Commuting the above identity with $x_1^+(0)$,we get

$$\begin{aligned} &q^{2l}\omega z(\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)(\varphi_1^+(0) - \varphi_1^-(0)) - q(\varphi_1^+(0) - \varphi_1^-(0))\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)) \\ &\qquad = q\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)\gamma^{-\frac{1}{2}}\varphi_1^+(1) - \gamma^{-\frac{1}{2}}\varphi_1^+(1)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z). \end{aligned}$$

From (2.7), we have

$$\begin{aligned} &\varphi_1^{\pm}(0) = q^{\pm h_1} \qquad \varphi_1^+(1) = (q - q^{-1})q^{h_1} a_1(1) \\ &\varphi_1^+(2) q^{-h_1} = (q - q^{-1}) a_1(2) + \frac{1}{2}(q - q^{-1})^2 a_1(1)^2 \\ &\varphi_1^+(3) q^{-h_1} = (q - q^{-1}) a_1(3) + (q - q^{-1})^2 a_1(1) a_1(2) + \frac{1}{6}(q - q^{-1})^3 a_1(1)^3 \end{aligned}$$

etc. So the identity

$$[a_1(1), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = q^{2l}\omega\gamma^{\frac{1}{2}} z \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)$$

can be given easily. Commuting repeatedly the formula

$$[X_i^+(0), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad i = 0, 1, \ldots, l$$

with $a_1(1)$, we get

$$[X_i^+(n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad \text{for } n > 0.$$

Similarly, we can also get

$$[X_i^+(-n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad \text{for } n > 0.$$

Therefore

$$[X_i^+(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0.$$

Commuting the identity (3.10) with $X_1^+(w)$, we get

$$\begin{cases} \dfrac{q^{2l} z\omega}{w}(\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)\widehat{\varphi}_1^+(w\gamma^{-\frac{1}{2}}) - q^2\widehat{\varphi}_1^+(w\gamma^{-\frac{1}{2}})\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)) \\ \quad = \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)\widehat{\varphi}_1^+(w\gamma^{-\frac{1}{2}}) - \widehat{\varphi}_1^+(w\gamma^{-\frac{1}{2}})\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) \\ \dfrac{q^{2l-2} z\omega}{w}(\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)\widehat{\varphi}_1^-(w\gamma^{\frac{1}{2}}) - \widehat{\varphi}_1^-(w\gamma^{\frac{1}{2}})\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)) \\ \quad = \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)\widehat{\varphi}_1^-(w\gamma^{\frac{1}{2}}) - q^{-2}\widehat{\varphi}_1^-(w\gamma^{\frac{1}{2}})\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) \end{cases} \tag{3.11}$$

where $\widehat{\varphi}_i^{\pm}(w) = q^{\mp h_i}\varphi_i^{\pm}(w)$. The above two formulae hold to any powers of w. Thus we obtain

$$\begin{cases} [a_1(n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = \dfrac{[n]}{n}(q^{2l+\frac{1}{2}}z\omega)^n \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) \\ [a_1(-n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = \dfrac{[n]}{n}(q^{2l-\frac{1}{2}}z\omega)^{-n} \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) \end{cases} \tag{3.12}$$

where $n \geqslant 0$. Meanwhile, from the intertwining relation, we also get

$$[e_i, \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad i \neq 0$$
$$[f_i, \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad i \neq 1.$$

Commuting the above two identities with $a_1(\pm n)$, we can easily prove

$$[x_2^+(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0.$$
$$[x_2^-(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0.$$

So we obtain

$$[a_2(n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0. \tag{3.13}$$

Then repeating the above procedures replacing $a_1(\pm n)$ by $a_2(\pm n)$, and so on, we get

$$[x_i^+(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \tag{3.14}$$
$$[x_i^-(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad i \neq 1. \tag{3.15}$$

We can conclude that

$$[\Psi(w), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \tag{3.16}$$
$$[a_i(\pm n), \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z)] = 0 \qquad i \neq 1. \tag{3.17}$$

Putting

$$a_1{}^*(k) = \sum_{n=1}^{l-1} \frac{([(l-n)k] - [(l-n-1)k])}{([lk] - [(l-1)k])[k]} a_n(k) + \frac{[k]}{([lk] - [(l-1)k])[2k]} a_l(k) \tag{3.18}$$

we get

$$[a_i(k), a_1{}^*(-k)] = -\delta_{i,1}\frac{[k]}{k}. \tag{3.19}$$

From the relations (3.16), (3.17), (3.12), (3.13), (3.6) and

$$q^d \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z) q^{-d} = \widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(q^{-1}z)$$

we obtain the type-I vertex operators

$$\widehat{\Psi}^{\Lambda_{1-i}V}_{\Lambda_i 2l+1}(z) = \exp\left[-\sum_{n=1}^{\infty}(q^{2l+\frac{1}{2}}z\omega)^n a_1{}^*(-n)\right] \exp\left[-\sum_{n=1}^{\infty}(q^{2l-\frac{1}{2}}z\omega)^{-n} a_1{}^*(n)\right]$$
$$\times e^{\lambda_1}(q^{2l}z\omega)^{\partial_{\lambda_1}+i}(-1)^{2\partial_{\lambda_l}+il} \tag{3.20}$$

where $i = 0, 1$, and

$$\widehat{\Psi}^{\Lambda_l V}_{\Lambda_l 2l+1}(z) = \omega^{-\frac{1}{2}} \exp\left[-\sum_{n=1}^{\infty} (q^{2l+\frac{1}{2}} z\omega)^n a_1{}^*(-n) \right]$$
$$\times \exp\left[-\sum_{n=1}^{\infty} (q^{2l-\frac{1}{2}} z\omega)^{-n} a_1{}^*(n) \right] e^{\lambda_1} (q^{2l} z\omega)^{\partial_{\lambda_1}+\frac{1}{2}} (-1)^{2\partial_{\lambda_l} - l(l+2)/2} \tag{3.21}$$

$$\begin{cases} \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} 2l+1-n}(z) = \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} 2l+2-n}(z) f_n - q f_n \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} 2l+2-n}(z) & n < l \\ \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} n}(z) = \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} n+1}(z) f_n - q f_n \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} n+1}(z) & n < l \\ \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l+1}(z) = \omega^{\frac{1}{2}} (\widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l+2}(z) f_l - q f_l \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l+2}(z)) \\ \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l}(z) = \omega^{\frac{1}{2}} (\widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l+1}(z) f_l - f_l \widehat{\Psi}^{\Lambda_{\sigma_2} V}_{\Lambda_{\sigma_1} l+1}(z)) \end{cases} \tag{3.22}$$

where we have used the normalization condition for $i = 0, 1, l$.

3.4. Vertex operator of type-II

For the vertex operator of type-II, the analogous commutation relations can be obtained. We have

$$[\Psi(W), \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)] = 0$$
$$[a_1(n), \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)] = -\frac{[n]}{n} (q^{\frac{1}{2}} \omega z)^n \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)$$
$$[a_1(-n), \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)] = -\frac{[n]}{n} (q^{-\frac{1}{2}} \omega z)^{-n} \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)$$
$$[a_i(\pm n), \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)] = 0 \qquad i \neq 1$$
$$[x_i^-, \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z)] = 0$$
$$q^{h_1} \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z) q^{-h_1} = q^{-1} \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_i 1}(z)$$
$$q^{h_i} \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z) q^{-h_i} = \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z) \qquad i \geqslant 2$$
$$q^{-d} \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(z) q^d = \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} 1}(qz).$$

So we can get the type-II vertex operators

$$\widehat{\Phi}^{V\Lambda_{1-i}}_{\Lambda_i 1}(z) = (q^{2l-1})^{i-1} \exp\left[\sum_{n=1}^{\infty} (q^{\frac{1}{2}} z\omega)^n a_1{}^*(-n) \right]$$
$$\times \exp\left[\sum_{n=1}^{\infty} (q^{-\frac{1}{2}} z\omega)^{-n} a_1{}^*(n) \right] e^{-\lambda_1} (z\omega)^{-\partial_{\lambda_1}+i} (-1)^{2\partial_{\lambda_l} - il}. \tag{3.23}$$

where $i = 0, 1$

$$\widehat{\Phi}^{V\Lambda_l}_{\Lambda_l 1}(z) = \omega^{-\frac{1}{2}} (-q)^{-l} \exp\left[\sum_{n=1}^{\infty} (q^{\frac{1}{2}} z\omega)^n a_1{}^*(-n) \right]$$
$$\times \exp\left[\sum_{n=1}^{\infty} (q^{-\frac{1}{2}} z\omega)^{-n} a_1{}^*(n) \right] e^{-\lambda_1} (z\omega)^{-\partial_{\lambda_1}+\frac{1}{2}} (-1)^{2\partial_{\lambda_l} - l^2/2} \tag{3.24}$$

$$\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} n+1}(z) = \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2} n}(z) e_n - q e_n \widehat{\Phi}^{V\Lambda_j}_{\Lambda_i n}(z) \qquad n < l$$

$$\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}2l+2-n}(z) = \widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}2l+1-n}(z)e_n - qe_n\widehat{\Phi}^{V\Lambda_j}_{\Lambda_i 2l+1-n}(z) \qquad n < l$$
$$\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+2}(z) = \omega^{\frac{1}{2}}(\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z)e_l - e_l\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z))$$
$$\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z) = \omega^{\frac{1}{2}}(\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l}(z)e_l - qe_l\widehat{\Phi}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l}(z)). \tag{3.25}$$

3.5. Commutation relations

In this section the commutation relations of vertex operators of type I will be calculated. For $U_q(B_l^{(1)})$, the intertwiner matrix R between $V_{Z_1} \otimes V_{Z_2}$ and $V_{Z_2} \otimes V_{Z_1}$ has been given in [17], but we prefer to use the form in [18] for a different form of coproduct. From the intertwiner property, we get

$$\left(\mathrm{id}\otimes r\left(\frac{z_2}{z_1}\right)R\left(\frac{z_2}{z_1}\right)\right)\left(\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}}(z_2)\otimes \mathrm{id}\right)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}}(z_1) = \left(\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}}(z_1)\otimes \mathrm{id}\right)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}}(z_2). \tag{3.26}$$

That is

$$\sum_{i,j=1}^{2l+1} r\left(\frac{z_2}{z_1}\right)R_{im,jn}\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}i}(z_2)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}j}(z_1)$$
$$= \widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}m}(z_1)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}n}(z_2) \qquad (m, n = 1, 2, \ldots, 2l+1).$$

In particular,

$$R_{i2l+1,j2l+1} = \begin{cases} 1 & i = j = 2l+1 \\ 0 & i \neq j \end{cases}$$

so

$$r\left(\frac{z_2}{z_1}\right)\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}2l+1}(z_2)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z_1) = \widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}2l+1}(z_1)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z_2). \tag{3.27}$$

By (3.18) and (3.19), we have

$$[a_1^*(k), a_1^*(r)] = -\delta_{k+r,0}\frac{[(l-1)k]-[(l-2)k]}{[lk]-[(l-1)k]}\frac{1}{k}.$$

Then

$$\left[-\sum_{n=1}^{\infty}(q^{2l-\frac{1}{2}}z_1\omega)^{-n}a_1^*(n), -\sum_{m=1}^{\infty}(q^{2l+\frac{1}{2}}z_2\omega)^{m}a_1^*(-m)\right]$$
$$= \ln\frac{(\xi z;\xi^2)_\infty(q^2z;\xi^2)_\infty}{(\xi^2z;\xi^2)_\infty(q^2\xi z;\xi^2)_\infty} \tag{3.28}$$

where $z = z_2/z_1$, $\xi = q^{2l-1}$ and $(z;p)_\infty := \prod_{n=0}^{\infty}(1-zp^n)$ and

$$\left[-\sum_{n=1}^{\infty}(q^{2l+\frac{1}{2}}z_1\omega)^{n}a_1^*(-n), -\sum_{m=1}^{\infty}(q^{2l-\frac{1}{2}}z_2\omega)^{-m}a_1^*(m)\right]$$
$$= \ln\frac{(\xi^2z^{-1};\xi^2)_\infty(q^2\xi z^{-1};\xi^2)_\infty}{(\xi z^{-1};\xi^2)_\infty(q^2z^{-1};\xi^2)_\infty}. \tag{3.29}$$

Note that

$$e^{\lambda_1}z_1{}^{\partial_{\lambda_1}}e^{\lambda_1}z_2{}^{\partial_{\lambda_1}} = \frac{z_1}{z_2}e^{\lambda_1}z_2{}^{\partial_{\lambda_1}}e^{\lambda_1}z_1{}^{\partial_{\lambda_1}}.$$

So we get

$$\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}2l+1}(z_1)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z_2) = z^{-1}\frac{(\xi z;\xi^2)_\infty(q^2z;\xi^2)_\infty(\xi^2z^{-1};\xi^2)_\infty(q^2\xi z^{-1};\xi^2)_\infty}{\xi^2z;\xi^2)_\infty(q^2\xi z;\xi^2)_\infty(\xi z^{-1};\xi^2)_\infty(q^2z^{-1};\xi^2)_\infty} \times\widehat{\Psi}^{\Lambda_{\sigma_1}V}_{\Lambda_{\sigma_2}2l+1}(z_2)\widehat{\Psi}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1}(z_1) \tag{3.30}$$

where $z = z_2/z_1$. Compare this with proposition 7.1 in [18]; the two results are the same.

3.6. Dual vertex operators

Define the intertwiners of the form

$$\widehat{\Psi^*}^{\Lambda_i}_{\Lambda_j V}(z) : V(\Lambda_j)\otimes V_z \to V(\Lambda_i)$$
$$\widehat{\Phi^*}^{\Lambda_i}_{V\Lambda_j}(z) : V_z\otimes V(\Lambda_j) \to V(\Lambda_i).$$

They are called dual vertex operators. Define their components by

$$\widehat{\Psi^*}^{\Lambda_i}_{\Lambda_j Vm}(z)\mid v\rangle = \widehat{\Psi^*}^{\Lambda_i}_{\Lambda_j V}(z)(|v\rangle\otimes v_m)$$
$$\widehat{\Phi^*}^{\Lambda_i}_{V\Lambda_j m}(z)\mid v\rangle = \widehat{\Psi^*}^{\Lambda_i}_{V\Lambda_j}(z)(v_m\otimes\mid v\rangle).$$

We impose the normalization

$$\langle\Lambda_1\mid\widehat{\Psi^*}^{\Lambda_1}_{\Lambda_0 V1}(z)\mid\Lambda_0\rangle = 1 \qquad \langle\Lambda_1\mid\widehat{\Phi^*}^{\Lambda_1}_{V\Lambda_0 1}(z)\mid\Lambda_0\rangle = 1$$
$$\langle\Lambda_0\mid\widehat{\Psi^*}^{\Lambda_0}_{\Lambda_1 V2l+1}(z)\mid\Lambda_1\rangle = 1 \qquad \langle\Lambda_0\mid\widehat{\Phi^*}^{\Lambda_0}_{V\Lambda_1 2l+1}(z)\mid\Lambda_1\rangle = 1$$
$$\langle\Lambda_l'\mid\widehat{\Psi^*}^{\Lambda_l'}_{\Lambda_l Vl+1}(z)\mid\Lambda_l\rangle = 1 \qquad \langle\Lambda_l'\mid\widehat{\Phi^*}^{\Lambda_l'}_{V\Lambda_l l+1}(z)\mid\Lambda_l\rangle = 1.$$

With the analogue discussion from [7, p 79], we get

$$\widehat{\Phi^*}^{\Lambda_{1-i}}_{V\Lambda_i 2l+1}(z) = q^{(2l-1)(1-i)}\hat{\Phi}^{V\Lambda_{1-i}}_{\Lambda_i 1}(q^{2l-1}z)$$
$$\widehat{\Psi^*}^{\Lambda_{1-i}}_{\Lambda_i V1}(z) = q^{(2l-1)i}\hat{\Psi}^{\Lambda_{1-i}V}_{\Lambda_i 2l+1}(q^{-2l+1}z)$$
$$\widehat{\Phi^*}^{\Lambda_l}_{V\Lambda_l 2l+1}(z) = (-q)^l\hat{\Phi}^{V\Lambda_l}_{\Lambda_l 1}(q^{2l-1}z)$$
$$\widehat{\Psi^*}^{\Lambda_l}_{\Lambda_l V1}(z) = (-q)^l\hat{\Psi}^{\Lambda_l V}_{\Lambda_l 2l+1}(q^{-2l+1}z)$$
$$\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+2-n}(z) = f_n\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1-n}(z) - q^{-1}\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}2l+1-n}(z)f_n \qquad n<l$$
$$\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}n+1}(z) = f_n\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}n}(z) - q^{-1}\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}n}(z)f_n \qquad n<l$$
$$\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+2}(z) = \omega^{\frac{1}{2}}(f_l\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z) - \widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z)f_l)$$
$$\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l+1}(z) = \omega^{\frac{1}{2}}(f_l\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z) - q^{-1}\widehat{\Psi^*}^{\Lambda_{\sigma_2}V}_{\Lambda_{\sigma_1}l}(z)f_l)$$
$$\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}n}(z) = e_n\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}n+1}(z) - q^{-1}\widehat{\Phi^*}^{V\Lambda_j}_{\Lambda_i n+1}(z)e_n \qquad n<l$$
$$\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}2l+1-n}(z) = e_n\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}2l+2-n}(z) - q^{-1}\widehat{\Phi^*}^{V\Lambda_j}_{\Lambda_i 2l+2-n}(z)e_n \qquad n<l$$
$$\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z) = \omega^{\frac{1}{2}}(e_l\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+2}(z) - q^{-1}\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+2}(z)e_l)$$
$$\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l}(z) = \omega^{\frac{1}{2}}(e_l\widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z) - \widehat{\Phi^*}^{V\Lambda_{\sigma_1}}_{\Lambda_{\sigma_2}l+1}(z)e_l).$$

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